Shall we buy and hold?

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Buy good mutual funds and hold on to them is touted by many investment advisors as a sound investment method. We will examine this method using both theoretical analysis and simulation on historical data.
Example: Three Funds

Let’s look at the performances of three mutual funds during a good year followed by a bad year in terms of percentage gain.

<table>
<thead>
<tr>
<th>Funds</th>
<th>%gain – Year1</th>
<th>%gain – Year2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>−50%</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
<td>−18%</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
<td>−5%</td>
</tr>
</tbody>
</table>

Now which one should we consider the best among the three?
Three Funds: Performances

Let us test with $1000 starting capital. Using all the capital \textbf{buy and hold} we have:

<table>
<thead>
<tr>
<th>Funds</th>
<th>%gain − Year1</th>
<th>%gain − Year2</th>
<th>Balances</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>−50%</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
<td>−18%</td>
<td>1066</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
<td>−5%</td>
<td>1054.5</td>
</tr>
</tbody>
</table>

\textbf{The Best:} Fund 2  
\textbf{The worst:} Fund 1
Three Funds: Performances

Next put **50%** of available capital in the fund and retain **50%** cash and re-balance at the beginning of the year:

<table>
<thead>
<tr>
<th>Funds</th>
<th>%gain – Year1</th>
<th>%gain – Year2</th>
<th>Balances</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>−50%</td>
<td>1125</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
<td>−18%</td>
<td>1046.5</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
<td>−5%</td>
<td>1028.6</td>
</tr>
</tbody>
</table>

The **Best**: Fund 1  
The **worst**: Fund 3
Lastly put **200%** of available capital (borrow 100%) in the fund and re-balance:

<table>
<thead>
<tr>
<th>Funds</th>
<th>%gain – Bull</th>
<th>%gain – Bear</th>
<th>Balances</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>−50%</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
<td>−18%</td>
<td>1024</td>
</tr>
<tr>
<td>3</td>
<td>11%</td>
<td>−5%</td>
<td>1098</td>
</tr>
</tbody>
</table>

**The Best:** Fund 3  
**The worst:** Fund 1  
The methods of investing matters.
In general, we denote the percentage investment size of the available capital by $s$ the gain in the good year by $g$ and the loss in the bad year by $l$. Then the average return per year $G(s)$ is

$$G(s) = (1 + sg)^{1/2}(1 + sl)^{1/2}$$ (1)

It is more convenient to use the return in log scale:

$$f(s) = \ln G(s) = \frac{1}{2} \ln(1 + sg) + \frac{1}{2} \ln(1 + sl).$$
Performance with variable sizes

Drawing the log return functions for Fund 1, Fund 2 and Fund 3 together we see that Fund 3 has the best potential and Fund 1 is the choice when margin is not available.

Figure 1: Log return functions
Test a general investment system

Key information

- The outcomes of the trades in terms of percentage gains
  \[ g_1 < g_2 < \ldots < g_N. \]
- The frequency \( p_n \) associated with the outcome \( g_n \).
- The size \( s \) of each trade as the percentage of the available capital.
- The total number of trades \( M \).

Then the total return is \( \Pi_{n=1}^{N} (1 + sg_n)^{M p_n} \) and the average rate of growth per trade is

\[ G(s) = \Pi_{n=1}^{N} (1 + sg_n)^{p_n}. \]
The log return function

Equivalently we can use the return in log scale:

\[ f(s) = \sum_{n=1}^{N} p_n \ln(1 + sg_n), \]

Here \( p_n \) is the probability for a trade to have a gain \( g_n \). We call this the log return function. Clearly \( G(s) = \exp(f(s)) \).

We define the efficiency index by

\[ \gamma = \max_{s \in \left(-\frac{1}{g_N}, -\frac{1}{g_1}\right)} f(s). \]
Remarks

- $\gamma < \infty$ iff $0 \in (g_1, g_N)$. This is the interesting case.
- $s > 1$ – on margins and $s < 0$ – shorts (hard to implement).
- $f$ is concave and, therefore, max of $f$ is always attained at an optimal investment size $\bar{s}$.
- $G = \exp(\gamma)$ is the average expected rate of growth per trade under the best investment size. The larger the $G$, the better.
- $\gamma > 0$ – system is potentially profitable.
- However, when $\bar{s} < 0$ it must be used in opposite direction.
Investment with Re-balance

Consider invest in a mutual fund at a fixed percentage $s$ of the total capital with say monthly re-balance. Consider $N$ months and denote the value of a share of the mutual fund at the beginning of each month by $v_n, n = 0, 1, \ldots, N$. We can regard this as an investment system with a profile of
\[
\{g_n = v_n/v_{n-1} - 1 : n = 1, 2, \ldots, N\}.
\]
Investment with Re-balance

Then the log return function and the exponential growth function of this investment system are

\[ f(s) = \frac{1}{N} \sum_{n=1}^{N} \ln(1 + s g_n), \]

and

\[ G(s) = \exp(f(s)) = \prod_{n=1}^{N} (1 + s g_n)^{1/N}, \]

respectively.
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respectively.

Clearly, \( s = 1 \) corresponding to the strategy of **buy and hold**.
Since most mutual funds underperform stock indices let us test a stock index to gain some insight.
Investing in stock indices

Example: SP500: Jan 1999 – Jan 2005

Best investment size: $\bar{s} = 35\%$.

Total percentage gain: 0.86\%
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This seems suggest that re-balancing is better than buy and hold.
Observe that

\[ G''(s) = \exp(f(s))[\left((f'(s))^2 + f''(s)\right)] \]

\[ = \exp(f(s))\left[\left(\sum p_n \frac{g_n}{1 + sg_n}\right)^2\right] \]

- \[ \sum p_n \left(\frac{g_n}{1 + sg_n}\right)^2 \] < 0

by convexity of \( x^2 \). The exponential growth function \( G(s) \) is concave.
Advantage of Re-balancing

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Historically DJIA has roughly the same growth rate compare to fixed income investment options measured by prime rate.
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Thus, re-balance with a fixed portion $s \in (0, 1)$ of the capital invested in DJIA is better than buy and hold this index.
The Impact of Frequency

Given $s \in (0, 1)$ what is the effect of adding a new re-balancing point in an existing re-balancing interval?
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Suppose the values of the index at the beginning and the end of the original re-balancing interval are $b$ and $e$, respectively.
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Suppose the values of the index at the beginning and the end of the original re-balancing interval are $b$ and $e$, respectively.

Adding a new re-balancing point and suppose that the value of the index at this point is $m$. 
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\[ d = \left( 1 + s\left( \frac{e}{b} - 1 \right) \right) \left( 1 + s\left( \frac{m}{b} - 1 \right) \right) \]

\[ - \left( 1 + s\left( \frac{e}{m} - 1 \right) \right) \left( 1 + s\left( \frac{m}{b} - 1 \right) \right) \]

\[ = \frac{s(1 - s)(e - m)(m - b)}{mb} \]
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\[
    d = \left( 1 + s\left(\frac{e}{b} - 1\right) \right) \\
    - \left( 1 + s\left(\frac{e}{m} - 1\right) \right) \left( 1 + s\left(\frac{m}{b} - 1\right) \right) \\
    = s(1-s)(e-m)(m-b) \quad \frac{mb}{mb}
\]

It follows that \( d > 0 \) when \( m \notin (b, e) \) and \( d < 0 \) when \( m \in (b, e) \).
Clearly we want to re-balance frequent enough to pick up the oscillations but not overly done to the extend that they interrupt the trends too many times.
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So let us test monthly re-balancing using DJIA.
References

References


Markowitz (1952), Portfolio Selection, The Journal of Finance, 8: 77-91.
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Samuelson (1979), Why we should not make mean log of wealth big though years to act are long, J. Banking and Finance, 3: 305-307.
Concluding remarks

- **Buy and hold** is not necessarily a sound investment strategy.
- In the study of portfolio theory the convex optimization method has not been fully explored, and
- the gap between theory and practice is still there.
THANK YOU!